Fourier transform of cos omega t

I was chatting with my colleague Steve Hanly about his recent post on the Fourier transform and power spectral density, and we thought it might help to go a bit more into the math and guts of the Fourier transform. As we know, the Fourier transform is a common and useful engineering tool for analyzing signals and vibrations, but sometimes it can produce some hard to interpret results. My aim for this post is to start things off with a refresher on the basics of the math behind the Fourier transformation, and lay the foundation for future posts that will go into more detail on how the Fourier transform should be used and interpreted. Complex Numbers: Magnitude, Angle, and Euler's Formula A complex number has separate real and imaginary components, such as the number 2 + j3. The letter j here is the imaginary number, which is equal to the square root of -1. I will use j as the imaginary number, as is more common in engineering, instead of the letter i, which is used in math and physics. Complex numbers have a magnitude: And an angle: A key property of complex numbers is called Euler's formula, which states: This exponential representation is very common with the Fourier transform. Note that an imaginary number of the format R + jI can be written as Aejξ where A is the magnitude and ξ is the angle. The Continuous Time Fourier Transform Continuous Fourier Equation The Fourier transform is defined by the equation And the inverse is These equations allow us to see what frequencies exist in the signal x(t). A more technical phrasing of this is to say these equations allow us to translate a signal between the time domain to the frequency domain. Note that these equations use a ξ (the Greek letter Xi) to imply frequency instead of ω (Omega) which generally refers to angular frequency (ω = 2πξ). The Fourier transform of a time dependent signal produces a frequency dependent function. A lot of engineers use omega because it is used in transfer functions, but here we are just looking at frequency. If we use the angular frequency instead of frequency, then we would have to apply a factor of 2π to either the transform or the inverse. The general rule is that the unit of the Fourier transform variable is the inverse of the original function's variable. Example Transformations Let's kick these equations around a bit. Let's try the super simple function x(t) = 2. Plugging this equation into the Fourier transform, we get: Integrals around infinity start to behave oddly, so this example will not be mathematically rigorous, but intuitively we can see that the integral of sin(ξt) and cos(ξt) as t goes from negative infinity to positive infinity should be 0, unless ξ = 0. At that point the equation simplified dramatically to: We can write the equation for X(ξ) using the Dirac delta function, δ(x), which is defined as: So, putting it all together, for x(t) = 2, X(ξ) = 2 δ(ξ). This means that the magnitude of X(ξ) is 0 everywhere except at ξ=0, where it is roughly 2∞. A more mathematically rigorous process, which you can find here, rests on the transform of the unit step function, which rests on the transform of an exponential decay. The purpose here is just to show that the transform of a DC signal will exist only at 0 Hz. Now let's look at the Fourier transform of a sine wave of frequency 1kHz. We can see that this is going to come to zero except for the case where ξ=±1000. At that point, using ξ=1000 for this example, the equation becomes: We can apply the trigonometric identity of sin(kt)cos(kt) = sin(2kt)/2 and sin2(kt) = (1-cos(2kt))/2, and we get: Similarly, at ξ=-1000, we will get: Using the Dirac function, we see that the Fourier transform of a 1kHz sine wave is: We can use the same methods to take the Fourier transform of cos(4000πt), and get: A few things jump out here. The first is that the Dirac function has an offset, which means we get the same spike that we saw for x(t) = 2, but this time we have spikes at the signal frequency and the negative of the signal frequency. This makes more sense when you remember that sin(-θ) = -sin(θ), and cos(-θ) = cos(θ). The second piece that should jump out is that the Fourier transform of the sine function is completely imaginary, while the cosine function only has real parts. This means that the angle of the transform of the sine function, which is the arctan of real over imaginary, is 90° off from the transform of the cosine, just like the sine and cosine functions are 90° off from each other. Convolution and Linearity The Fourier transform is linear, meaning that the transform of Ax(t) + By(t) is AX(ξ) + BY(ξ), where A and B are constants, and X and Y are the transforms of x and y. This property may seem obvious, but it needs to be explicitly stated because it underpins many of the uses of the transform, which I'll get to later. One property the Fourier transform does not have is that the transform of the product of functions is not the same as the product of the transforms. Or, stated more simply: The Fourier transform of the product of two signals is the convolution of the two signals, which is noted by an asterix (*), and defined as: This is a bit complicated, so let's try this out. We'll take the Fourier transform of cos(1000πt)cos(3000πt). We know the transform of a cosine, so we can use convolution to see that we should get: A long equation, but since δ(x) = 0 where x ≠0, we can see that there are only a few spots where we get non-zero answers. For example, in the first term we have: For δ(ξ'-500) to be non-zero, we need to set ξ'=500. Then we have δ(ξ - ξ' - 1500) = δ(ξ – 500 - 1500) = δ(ξ - 2000), so at frequency 2000 we have a value of δ(0)/2. We can continue this for all 4 terms and see that we get the same result at frequency 2000, 1000, -1000, and -2000. What happened here? We multiplied two frequencies together and the result is that we essentially re-centered the response for one at the frequency of the other. cos(1000πt) gives spikes at 500 and -500 Hz, cos(3000πt) works at 1500 and -1500 Hz, so we took the ±500 Hz spikes and re-centered them at 1500 and -1500 Hz. Similarly, if we take any waveform and multiply it by a sine or cosine, the transform of the resulting signal is the original re-centered at the frequencies of the sine wave. This is a very important feature that can easily get confusing, so I'll cover it more in future posts. The Discrete Time Fourier Transform How to Use the Discrete Fourier Transform The discrete Fourier transform (DFT) is the most direct way to apply the Fourier transform. To use it, you just sample some data points, apply the equation, and analyze the results. Sampling a signal takes it from the continuous time domain into discrete time. Here, I'll use square brackets, [], instead of parentheses, (), to show discrete vs continuous time functions. The input to a discrete function must be a whole number. Sampling a signal x(t) at sample period Ts will turn it into x[n] such that: x[n]=x(nTs) So a sine wave of frequency fs = 1kHz sampled at Ts = 10uS will become: x[n] = sin(2πfsnTs) = sin(0.02πn) I'm using n instead of t to denote that we are just looking at a sequence of points; there is no inherent time signature here. A 100 Hz sine wave sampled at 1kHz looks exactly like a 10 Hz sine wave sampled at 100 Hz. Discrete Fourier Equation A common form of the DFT equation is Here, N is the total number of points included in the equation, and m, the input to the transformed function, roughly correlates to the frequency in the same way that n correlates to time with the equation: Let's say we're gathering 10 samples sampled at 100uS. For an input signal of x(t)=2, we will get x(0) = 2, x(1) = 2, x(2) = 2, etc. Plugging this into the DFT, we get: n Real Results Imaginary Results 0 1 - j0 1 + 0.809017 - j0.587785 2 + 0.309017 - j0.951057 3 - 0.30902 - j0.951057 4 - 0.80902 - j0.587785 5 - 1 - j0 6 - 0.80902 + j0.58779 7 - 0.30902 + j0.95106 8 + 0.309017 + j0.95106 9 + 0.809017 + j0.58779 X[1] = 0 - j0 As we expect, the sum of a sine and cosine over the whole period equals 0. We can assume that all other values of m will similarly be 0, so, similar to the time domain example, we have: To get the actual signal magnitude we need to divide by N. Let's make things more complicated and take the DFT of the following signal, using 10 points sampled at 100 uS: Running this through the DFT equation will get us the following values: The concept of negative and positive frequency can be as simple as a wheel rotating one way or the other way: a signed value of frequency can indicate both the rate and direction of rotation. The rate is expressed in units such as revolutions (a.k.a. cycles) per second (hertz) or radian/second (where 1 cycle corresponds to 2π radians). Sinusoids Let ω be a nonnegative parameter with units of radians/second. Then the angular function (angle vs. time) −ωt + θ, has slope −ω, which is called a negative frequency. But when the function is used as the argument of a cosine operator, the result is indistinguishable from cos(ωt − θ). Similarly, sin(−ωt + θ) is indistinguishable from sin(ωt − θ + π). Thus any sinusoid can be represented in terms of positive frequencies. The sign of the underlying phase slope is ambiguous. A negative frequency causes the sin function (violet) to lead the cos (red) by 1/4 cycle. The vector (cos t, sin t) rotates counter-clockwise at 1 radian/second, and completes a circle every 2π seconds. The vector (cos −t, sin −t) rotates in the other direction (not shown). The ambiguity is resolved when the cosine and sine operators can be observed simultaneously, because cos(ωt + θ) leads sin(ωt + θ) by 1/4 cycle (= π/2 radians) when ω > 0, and lags by 1/4 cycle when ω < 0. Similarly, a vector, (cos t, sin t), rotates counter-clockwise at 1 radian/second, and completes a circle every 2π seconds, and the vector (cos −t, sin −t) rotates in the other direction. The sign of ω is also preserved in the complex-valued function: e i ω t = cos (ω t) R (t) + i ⋅ sin (ω t) I (t) , {\displaystyle e^{i\omega t}=\underbrace {\cos(\omega t)} _{R(t)}+i\cdot \underbrace {\sin(\omega t)} _{I(t)},} [A] (Eq.1) since R(t) and I(t) can be separately extracted and compared. Although e i ω t {\displaystyle e^{i\omega t}}   clearly contains more information than either of its components, a common interpretation is that it is a simpler function, because:It simplifies many important trigonometric calculations, which leads to its formal description as the analytic representation of   cos (ω t) {\displaystyle \cos(\omega t)} .[B]A corollary of Eq.1 is: cos (ω t) = 1 2 (e i ω t + e − i ω t) , {\displaystyle \cos(\omega t)={\begin{matrix}{\frac {1}{2}}\end{matrix}}\left(e^{i\omega t}+e^{-i\omega t}\right),} (Eq.2) which gives rise to the interpretation that cos(ωt) comprises both positive and negative frequencies. But the sum is actually a cancellation that contains less, not more, information. Any measure that indicates both frequencies includes a false positive (or alias), because ω can have only one sign.[C] The Fourier transform, for instance, merely tells us that cos(ωt) cross-correlates equally well with cos(ωt) + i sin(ωt) as with cos(ωt) − i sin(ωt).[D] Applications Perhaps the most well-known application of negative frequency is the calculation: X (ω) = ∫ a b x (t) ⋅ e − i ω t d t , {\displaystyle X(\omega)=\int _{a}^{b}x(t)\cdot e^{-i\omega t}dt,} which is a measure of the amount of frequency ω in the function x(t) over the interval (a, b). When evaluated as a continuous function of ω for the theoretical interval (−∞, ∞), it is known as the Fourier transform of x(t). A brief explanation is that the product of two complex sinusoids is also a complex sinusoid whose frequency is the sum of the original frequencies. So when ω is positive, e − i ω t {\displaystyle e^{-i\omega t}} causes all the frequencies of x(t) to be reduced by amount ω. Whatever part of x(t) that was at frequency ω is changed to frequency zero, which is just a constant whose amplitude level is a measure of the strength of the original ω content. And whatever part of x(t) that was at frequency zero is changed to a sinusoid at frequency −ω. Similarly, all other frequencies are changed to non-zero values. As the interval (a, b) increases, the contribution of the constant term grows in proportion. But the contributions of the sinusoidal terms only oscillate around zero. So X(ω) improves as a relative measure of the amount of frequency ω in the function x(t). The Fourier transform of   e i ω t {\displaystyle e^{i\omega t}}   produces a non-zero response only at frequency ω. The transform of cos (ω t) {\displaystyle \cos(\omega t)} has responses at both ω and −ω, as anticipated by Eq.2. Sampling of positive and negative frequencies and aliasing Main article: Aliasing § Complex sinusoids This figure depicts two complex sinusoids, colored gold and cyan, that fit the same sets of real and imaginary sample points. They are thus aliases of each other when sampled at the rate (fs) indicated by the grid lines. The gold-colored function depicts a positive frequency, because its real part (the cos function) leads its imaginary part by 1/4 of one cycle. The cyan function depicts a negative frequency, because its real part lags the imaginary part. Notes ^ The equivalence is called Euler's formula ^ See Euler's formula § Relationship to trigonometry and Phasor § Addition for examples of calculations simplified by the complex representation. ^ Conversely, any measure that indicates only one frequency has made an assumption, perhaps based on collateral information. ^ cos(ωt) and sin(ωt) are orthogonal functions, so the imaginary parts of both correlations are zero. Further reading Positive and Negative Frequencies Lyons, Richard G. (Nov 11, 2010). Chapt 8.4. Understanding Digital Signal Processing (3rd ed.). Prentice Hall. 944 pgs. ISBN 0137027419. Retrieved from "